Connections: (here: $\mathbb{C}$-vect bundles, also for $\mathbb{R}$-bundles!)

$E \to M$ smooth $\mathbb{C}$-vector bundle over a smooth manifold M

$C^\infty(E) :=$ smooth sections of E (in local triv., $\simeq C^\infty(U_i, \mathbb{C}^r)$)

$\Lambda^k(E) :=$ smooth k-forms in E = sections of $\Lambda^k T^*M \otimes E$ (loc. $\simeq \Lambda^k(U_i, \mathbb{C}^r)$).

Def: A connection on E is a $\mathbb{C}$-linear map (in fact, sheaf hom.: also for $E_{|U}$ $\forall U \subset M$ open).

$$\underset{\text{sections}}{\nabla : C^\infty(E)} \longrightarrow \Lambda^1(E) = \underset{\text{1-forms with values in } E}{C^\infty(T^*M \otimes E)}$$

st. $\nabla(fs) = df \otimes s + f \nabla s \quad \forall f \in C^\infty(M, \mathbb{C}), \forall s$ section of E

("covariant derivative": $v \in T_xM \Rightarrow \nabla_v s(x)$ "derivative" of s in dir^n of v not defined canonically! since fibers of E are not can. $\cong$ each other)

Prop: Connections form an affine space modelled on $\Lambda^1(M, \mathrm{End}\, E)$ 1-forms w/ values in $\mathrm{End}(E)$.

Pf: $\nabla'(fs) - \nabla(fs) = f(\nabla' s - \nabla s)$, so $\nabla' - \nabla$ is $C^\infty(M)$-linear (a tensor, not a diff op!) & hence can show $\nabla' - \nabla =: a \in \Lambda^1(M, \mathrm{End}\, E)$ 1-form in endoms of E.

Conversely, ∇ conn., $a \in \Lambda^1(M, \mathrm{End}\, E) \Rightarrow \nabla + a$ also satisfies Leibniz rule ▲

- Connection 1-form:

In a local trivialization, $E_{|U} \simeq U \times \mathbb{C}^r$, then sections $\leftrightarrow$ $\mathbb{C}^r$-valued f^{ns} and $\exists$ "trivial connection" d = componentwise differentiation

Any other conn. is then $\nabla = d + A$, $A \in \Lambda^1(U, \mathrm{End}\, E)$.
matrix-valued 1-form.

Change of trivialization: a same section s is given by

$s \in C^\infty(U, \mathbb{C}^r) \leftrightarrow s' = \varphi \cdot s \in C^\infty(U, \mathbb{C}^r) \qquad \varphi \in C^\infty(U, GL(r, \mathbb{C}))$.

for ∇s: $(d + A)s \longleftrightarrow (d + A')s' = \varphi \cdot [(d + A)s]$

so $\varphi^{-1}(d + A')\varphi = d + A$

ie. $A = \varphi^{-1} A' \varphi + \varphi^{-1} d\varphi \in \Lambda^1(U, \mathrm{End}(\mathbb{C}^r))$.

So... $\nabla = d + A$ loc, but A transforms in a non-obvious manner.

- ∇_1 on E_1, ∇_2 on E_2 induce connections on

$$E_1 \oplus E_2 : \nabla(s_1 \oplus s_2) = \nabla_1 s_1 \oplus \nabla_2 s_2$$
$$E_1 \otimes E_2 : \nabla(s_1 \otimes s_2) = \nabla_1 s_1 \otimes s_2 + s_1 \otimes \nabla_2 s_2$$
$$\mathrm{Hom}(E_1, E_2) : (\nabla f)(s_1) = \nabla_2(f(s_1)) - \underbrace{f(\nabla_1 s_1)}_{\text{apply } f \text{ to 1-form in } E_1 \text{ by}}.$$

$f(\alpha \otimes s) := \alpha \otimes f(s)$

- $\varphi : N \to M$, ∇ on $E \to M$ induces $(\varphi^*\nabla)$ on $\varphi^* E$: $s \in C^\infty(E) \Rightarrow (\varphi^*\nabla)(s \circ \varphi) = \nabla s \circ \varphi$
  in local trivs: $\nabla = d + A$ on $U \Rightarrow \varphi^*\nabla = d + \varphi^* A$ in induced triv^n on $\varphi^{-1}(U)$

* $(E, \langle\cdot,\cdot\rangle)$ herm. vect bundle $\Rightarrow$ say ∇ on E is <u>hermitian</u> if

$$d(\langle s_1, s_2\rangle) = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle \quad \forall s_1, s_2 \qquad (\text{here } \langle \alpha\otimes s, s'\rangle := \langle s, s'\rangle \alpha \ \ldots)$$

(i.e. Leibniz rule holds for $\langle\cdot,\cdot\rangle$)

In a hermitian local trivialization (in which $\langle\cdot,\cdot\rangle$ = std Herm. metric on $\mathbb{C}^r$), $\nabla = d + A$ is a Hermitian connection iff A is <u>antihermitian</u> $(A^t = -\bar{A})$

<u>**Curvature:**</u> ∇ connection on $E \to M \Rightarrow F_\nabla \in \Lambda^2(M, \mathrm{End}\, E)$

(Riemann curvature = special case of Levi-Civita conn. on TM).

- $\nabla : C^\infty(E) \to \Lambda^1(E)$ determines $d^\nabla : \Lambda^k(E) \to \Lambda^{k+1}(E)$

$$d^\nabla(\underbrace{\alpha}_{k\text{-form}} \otimes \underbrace{s}_{\text{sec}^n \text{ of } E}) = (d\alpha)\otimes s + (-1)^k \alpha \wedge \nabla s.$$

- check: d^∇ satisfies Leibniz rule $d^\nabla(\beta \wedge \tau) = d\beta \wedge \tau + (-1)^{|\beta|} \beta \wedge d^\nabla \tau$.

<u>Def/Prop:</u> $F_\nabla := d^\nabla \circ d^\nabla : C^\infty(E) \to \Lambda^2(E)$ is $C^\infty(M)$-linear, hence a tensor $\in \Lambda^2(M, \mathrm{End}\, E)$.
<u>curvature of ∇</u>

(in other terms, $F_\nabla(f \cdot s) = f\, F_\nabla(s) \quad \forall f \in C^\infty(M)$)

* In local trivialization: $\nabla = d + A \qquad A$ = matrix-valued 1-form
  $\Rightarrow d^\nabla = d + A\wedge\cdot \qquad$ (matrix of 1-forms $\wedge$ vector of k-forms)

$$\rightsquigarrow F_\nabla(s) = (d+A)(d+A)s = d^2(s) + d(As) + A\wedge ds + A \wedge As$$
$$= (dA)s + (A\wedge A)s \qquad \text{(indeed } C^\infty\text{-linear)}$$

ie. <u>loc. $F_\nabla = dA + A\wedge A$</u> (matrix of 2-forms). (check: $A \mapsto \varphi^{-1}A\varphi + \varphi^{-1}d\varphi$
$\Rightarrow F \mapsto \varphi^{-1}F\varphi$ ✓)

Properties of curvature:

- Biangi identity: $\| \; d^{\nabla}(F_\nabla) = 0$ where $F_\nabla \in \Lambda^2(M, \operatorname{End} E)$
  $d^\nabla : \Lambda^2(M, \operatorname{End} E) \to \Lambda^3(M, \operatorname{End} E)$
  for conn. on $\operatorname{End}(E)$ induced by ∇!

  Pf: direct calc. in trivialization: $\nabla = d + A$ induces $\nabla^{\mathrm{End}} = d + [A, \cdot]$
  $\Rightarrow$ this says $dF_\nabla + A \wedge F_\nabla - F_\nabla \wedge A = 0$, easily checked.

- curvature behaves naturally wrt $\oplus$ $\otimes$ etc., for instance
  $F_{\nabla_1 \oplus \nabla_2} = F_{\nabla_1} \oplus F_{\nabla_2}$ $\quad F_{\nabla_1 \otimes 1 + 1 \otimes \nabla_2} = F_{\nabla_1} \otimes 1 + 1 \otimes F_{\nabla_2}$
  and pullback $F_{\varphi^*\nabla} = \varphi^* F_\nabla$. ($\varphi : N \to M$ induces $\varphi^* : \Lambda^2(M, \operatorname{End} E) \to \Lambda^2(N, \operatorname{End} \varphi^* E)$)
- E hermitian, ∇ Hermitian connection $\Rightarrow$ F_∇ takes values in antihermitian endom's

  Pf: in hermitian loc. trivialization, $\nabla = d + A$, A antiherm. matrix of 1-forms $(A^t = -\bar{A})$
  $\rightsquigarrow F_\nabla = dA + A \wedge A$ satisfies $F_\nabla^t = -\overline{F_\nabla}$
  (using: $(A_1 \wedge A_2)^t = -A_2^t \wedge A_1^t$ for matrices of 1-forms vs. $(M_1 M_2)^t = M_2^t M_1^t$ for matrices of 0-forms)

Chern classes: diff. geom. construction of $c_k(E) \in H^{2k}_{dR}(M, \mathbb{R})$

(in De Rham cohomology $H^j_{dR} = \dfrac{\ker(d : \Lambda^j \to \Lambda^{j+1}(M))}{\operatorname{Im}(d : \Lambda^{j-1} \to \Lambda^j)}$ closed diff forms mod exact forms)

(Thm: $H^j_{dR}(M, \mathbb{R}) \simeq H^j_{sing}(M, \mathbb{R})$; not clear how to see $c_k(E) \in H^{2k}(M, \mathbb{Z})$ using this approach ...)

Easiest: line bundle $L \to M$

- Choose a connection $\nabla \rightsquigarrow F_\nabla$ curvature $\in \Lambda^2(M, \operatorname{End}(L)) = \Lambda^2(M, \mathbb{C})$
  locally $\nabla = d + A$, $A \in \Lambda^1(U, \mathbb{C})$ and $F_\nabla = dA + A \wedge A = dA \Rightarrow F_\nabla$ is closed.
  So: $[F_\nabla] \in H^2(M, \mathbb{C})$. Set $c_1(L) = \frac{i}{2\pi}[F_\nabla]$.
- indep^{ce} of ∇: $\nabla' = \nabla + a$, $a \in \Lambda^1(M, \mathbb{C}) \rightsquigarrow F_{\nabla'} = F_\nabla + da$, $[F_{\nabla'}] = [F_\nabla]$
- picking ∇ compat. w/ some herm. structure, F_∇ antiherm. ie. pure imaginary
  $\rightsquigarrow c_1(L) \in H^2(M, \mathbb{R})$ ✓

In higher rank, $F_\nabla = dA + A \wedge A$ isn't closed anymore, but eg. $\operatorname{tr}(F_\nabla) \in \Lambda^2(M, \mathbb{C})$ is still closed $\left(\operatorname{tr} A \wedge A = \sum_i (A \wedge A)_{ii} = \sum_{ij} A_{ij} \wedge A_{ji} = 0 \quad (i,j \text{ cancels } j,i)\right)$.

<u>In general:</u> need a conjugation-invariant homogeneous polynomial expression in the entries of the matrix of 2-forms F_∇.

Total Chern form: $c(E,\nabla) = \det\Big(\underbrace{\mathrm{Id} + \tfrac{i}{2\pi} F_\nabla}_{\text{in loc. triv.;}}\Big) \in \Lambda^{\mathrm{even}}(M,\mathbb{C})$

$r\times r$ matrix with entries = 0-forms + 2-forms,
formula for det in terms of entries gives
0-form + 2-form + ... + $(2r)$-form.

- Inv$^{\underline{ce}}$ of det under conjugation $\Rightarrow$ this is well-def$^{\underline{d}}$ (indep^{t} of loc. trivialization used to compute det)
- Bianchi identity (or verification by hand) $\Rightarrow$ $c(E,\nabla)$ is a closed form.

$\Rightarrow c(E) := \left[\det\left(\mathrm{Id} + \tfrac{i}{2\pi} F_\nabla\right)\right] \in H^{\mathrm{even}}_{dR}(M,\mathbb{C})$ indep^t of ∇. (theorem).

<u>general construction</u> (Chern-Weil theory)

* let $P : gl(r,\mathbb{C}) \times \dots \times gl(r,\mathbb{C}) \to \mathbb{C}$ symmetric, k-multilinear, conj. inv^t i.e. $P(QB_1Q^{-1}, \dots, QB_kQ^{-1}) = P(B_1, \dots, B_k)$.

$\rightsquigarrow$ defines $P : \Lambda^2(M,\mathrm{End}\,E) \times \dots \times \Lambda^2(M,\mathrm{End}\,E) \to \Lambda^{2k}(M,\mathbb{C})$ by
$P(\alpha_1 \otimes B_1, \dots, \alpha_k \otimes B_k) = (\alpha_1 \wedge \dots \wedge \alpha_k)\, P(B_1, \dots, B_k)$

also for forms of arbitrary degrees!

$r\times r$ matrices once triv$^{\underline{n}}$ chosen; conj. inv$^{\underline{ce}}$ $\Rightarrow$ indep^t of triv$^{\underline{n}}$.

<u>Thm:</u> $P(F_\nabla \dots F_\nabla)$ is closed and $[P(F_\nabla)] \in H^{2k}(M,\mathbb{C})$ is independent of the connection ∇.

<u>Lemma:</u> $\forall \gamma_1 \dots \gamma_k \in \Lambda^*(M,\mathrm{End}\,E)$, $\quad d(P(\gamma_1 \dots \gamma_k)) = \sum_{j=1}^{k} \pm P(\gamma_1, \dots, d^{\nabla^{\mathrm{End}}}(\gamma_j), \dots, \gamma_k)$

<u>Pf:</u> in local triv$^{\underline{n}}$ $\nabla = d + A$, $\nabla^{\mathrm{End}} = d + [A,\cdot]$ so

$$\mathrm{RHS} = \underbrace{\sum \pm P(\gamma_1, \dots, d\gamma_j, \dots, \gamma_k)}_{d(P(\gamma_1 \dots \gamma_k))} + \underbrace{\sum \pm P(\gamma_1, \dots, [A,\gamma_j], \dots, \gamma_k)}_{= \frac{d}{dt} P(e^{tA}\gamma_1 e^{-tA}, \dots, e^{tA}\gamma_k e^{-tA}) = 0 \ \text{(inv}^{\text{ce}}\text{)}}$$ $\blacktriangle$

Pf. thm: • Bianchi identity $d^{\nabla^{End}}(F_\nabla)=0 \Rightarrow$ by lemma, $dP(F_\nabla,\dots,F_\nabla)=0$

• for $a\in\Lambda^1(M,\mathrm{End}E)$, $\frac{d}{dt}_{|t=0} F_{\nabla+ta} = d^{\nabla^{End}} a$ (differentiate $dA+[A,A]$ wrt A)

so $\frac{d}{dt}_{|t=0} P(F_{\nabla+ta}\dots F_{\nabla+ta}) = \sum P(F_\nabla,\dots,d^{\nabla^{End}}a,\dots,F_\nabla)$

$\underset{\text{lemma + Bianchi}}{=} \sum \pm\, d\big(P(F_\nabla,\dots,a,\dots,F_\nabla)\big)$ is exact. ∆

Apply to:

• <u>Chern classes</u>: $c(E) := \left[\det\left(Id+\frac{i}{2\pi}F_\nabla\right)\right] = 1+\frac{i}{2\pi}[\mathrm{tr}\,F_\nabla]+\dots+\left(\frac{i}{2\pi}\right)^r[\det F_\nabla]$

$E\to M$ rank r $\mathbb{C}$ v.b.

$= 1+c_1(E)+\dots+c_r(E)$, $c_k(E)\in H^{2k}(M,\mathbb{C})$

In fact, by considering a hermitian connection, $\overline{F_\nabla}=-F_\nabla^t \Rightarrow c_k(E)\in H^{2k}(M,\mathbb{R})$.

(can also define Euler class, Pontrjagin classes, Chern character... in this way).

To compare w/ alg. top definition (under $H^*(M,\mathbb{Z})\to H^*(M,\mathbb{R})$): check axioms!

• <u>Prop</u>: $c(E_1\oplus E_2) = c(E_1)\wedge c(E_2)$

<u>Pf.</u> picking ∇_1,∇_2: $Id+\frac{i}{2\pi}F_{\nabla_1\oplus\nabla_2} = \left[\begin{array}{c|c} Id+\frac{i}{2\pi}F_{\nabla_1} & 0 \\ \hline 0 & Id+\frac{i}{2\pi}F_{\nabla_2}\end{array}\right] \Rightarrow \det = \det_1 \det_2$ ∆

• <u>Prop</u>: $\varphi: N\to M$, $E\to M$ cx. vect bundle, then $c(\varphi^*E)=\varphi^*c(E)$ in $H^*(N)$

<u>Pf.</u> calc: pullback connection $\varphi^*\nabla$ (loc. $\varphi^*(d+A) = (d+\varphi^*A)$) has $F_{\varphi^*\nabla}=\varphi^*F_\nabla$.

• On the other hand, the normalization axiom, eg $c_1(\tau\to\mathbb{C}P^1) = $ gen. of H^2, requires a nontrivial calculation of curvature of a specific connection on τ...